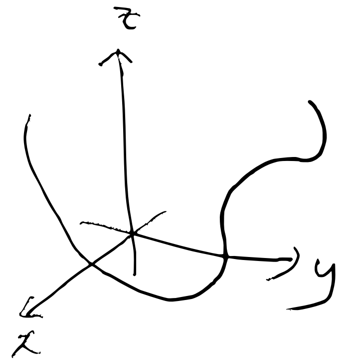
What is a "curve"? Mathematically it means one-dimensional.

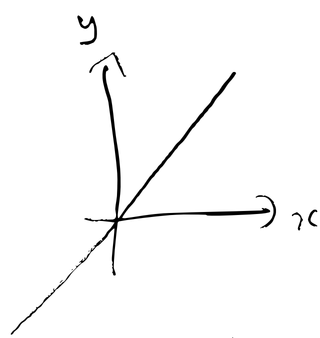


The concept of dimension is intuitively easy.
We're interested in using equations to express curves in 2D or 3D.

We are much more familiar with curves in 2D.

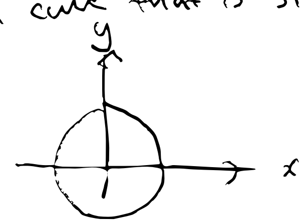
<u>Example</u>

- The graph of $y=x$ is



we call it a "line", namely a curve that is straight.

- The graph of $x^2+y^2=1$ is



the circle centered at the origin $(0,0)$ with radius 1.

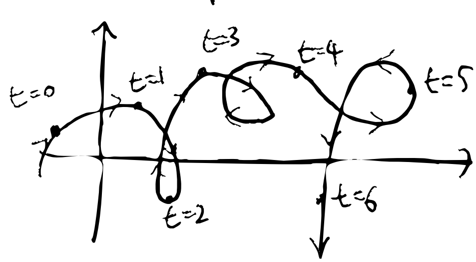
These equations are implicit, meaning that you gather the points that satisfy this equation.

In some sense it is passive, because of the equation gets more complicated, it's almost impossible to showcase a point that satisfies the equation. Namely, can you find a point (x, y) that satisfies

$$x^7 - 2.5x^6y^2 + 3x^3\sin y - 79x^{79}y^{105} + 62945 = 0?$$

To me it sounds impossible.

There is on the other hand more "explicit" expression of the curves, called parametric curves. The idea is that a curve is the trajectory of a particle moving around, so you can express the curve in terms of the position of the particle at a given time.

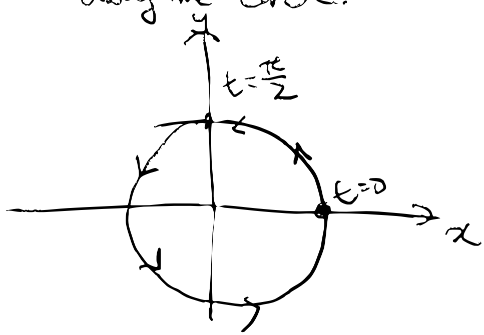


So you express x, y in terms of a new variable t (time).

<u>Example</u> Consider expressing the circle $x^2+y^2=1$ as a parametric curve.

$$x = \cos t$$
$$y = \sin t$$

Then the particle is at $(1,0)$ at $t=0$, and moves anticlockwise along the circle.

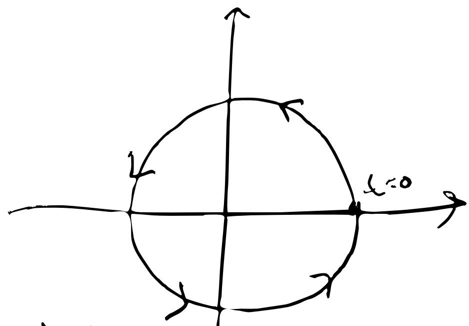


<u>Note</u> The parametric curve is, to be precise, about the motion of a particle, and is something more than a trajectory. For example, the same trajectory can be traversed by different kinds of motions.

For example, the same circle $x^2+y^2=1$ can be travelled by a particle that is as twice fast as the previous example.

"sped-up"

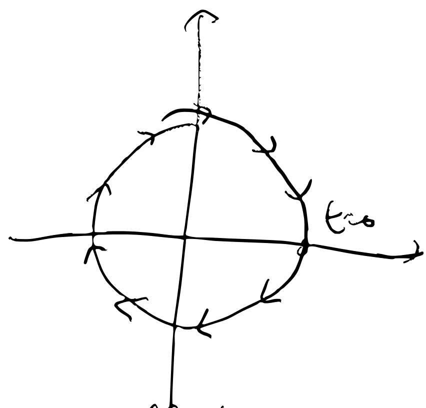
$$x=\cos(2t)$$
$$y=\sin(2t)$$



Or, you could run them clockwise.

"clockwise"

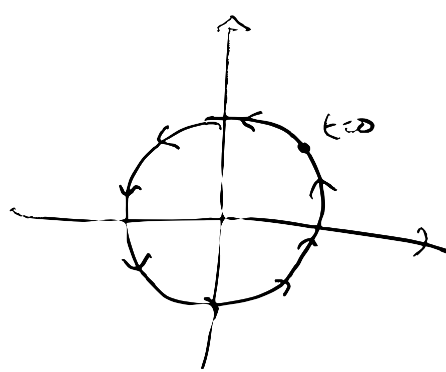
$$x=\cos(-t)$$
$$y=\sin(-t)$$



Or, you could start at a different spot.

"time shift"

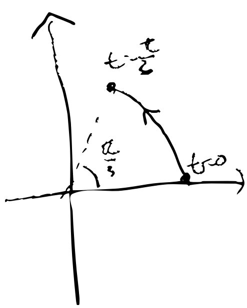
$$x=\cos\left(t+\frac{\pi}{4}\right)$$
$$y=\sin\left(t+\frac{\pi}{4}\right)$$



These kinds of change in motion, which are obtained by manipulating the time, are called **reparametrization**.

Parametric curves can also express a part of the curve easily. A part of a curve is called a **segment**.

Example $x = \cos t$, $y = \sin t$, $0 \le t \le \frac{\pi}{3}$ is a segment.



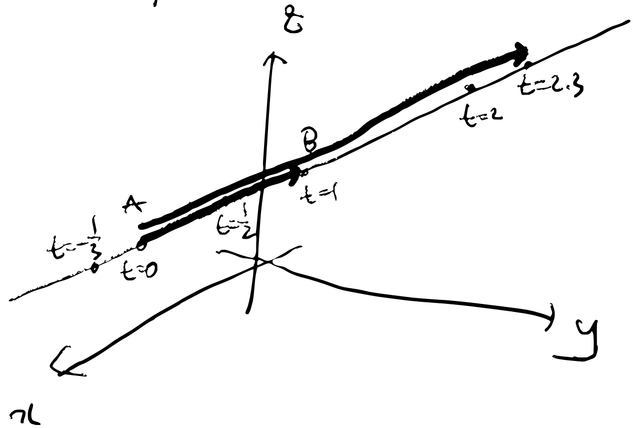
We say $(1,0)$ is the **initial point** and $(\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))$ is the **terminal point**.

We now list some of the most basic types of curves in 2D and 3D.

Lines Using vectors, it is now easy to find the parametric equation for a line that passes through the two given points.

Suppose we want to know the parametric curve that passes through the point $A=(a_1,a_2,a_3)$ at $t=0$ and the point $B=(b_1,b_2,b_3)$ at $t=1$.

Recall that $\overrightarrow{AB}=\langle b_1-a_1,\ b_2-a_2,\ b_3-a_3\rangle$. Then, the position of the particle at time say $t=2.3$ is the point P such that $\overrightarrow{AP}=2.3\overrightarrow{AB}$!



In this case, $\overrightarrow{AB}=\langle b_1-a_1,\ b_2-a_2,\ b_3-a_3\rangle$ is called the direction vector.

This means the parametric equation is

$$\boxed{\begin{aligned} x &= (b_1-a_1)t+a_1 \\ y &= (b_2-a_2)t+a_2 \\ z &= (b_3-a_3)t+a_3. \end{aligned}}$$

3D line passing through (a_1,a_2,a_3) and (b_1,b_2,b_3)

$$\boxed{\begin{aligned} x &= (b_1-a_1)t+a_1 \\ y &= (b_2-a_2)t+a_2 \end{aligned}}$$

2D line passing through (a_1,a_2) and (b_1,b_2)

$$\boxed{\begin{aligned} x &= v_1t+a_1 \\ y &= v_2t+a_2 \\ z &= v_3t+a_3 \end{aligned}}$$

3D line through (a_1,a_2,a_3) and parallel to $\langle v_1,v_2,v_3\rangle$.

For 2D lines, you can eliminate t to obtain the implicit equation for the line. Namely,

$$x = (b_1 - a_1)t + a_1$$

$$\downarrow \times \frac{b_2 - a_2}{b_1 - a_1}$$

$$\frac{b_2 - a_2}{b_1 - a_1} x = (b_2 - a_2)t + a_1 \frac{b_2 - a_2}{b_1 - a_1}$$

$$\downarrow - \left(y = (b_2 - a_2)t + a_2\right)$$

$$\frac{b_2 - a_2}{b_1 - a_1} x - y = \frac{a_1(b_2 - a_2)}{b_1 - a_1} - a_2 = \frac{a_1 b_2 - a_1 a_2 - b_1 a_2 + a_1 a_2}{b_1 - a_1}$$

$$= \frac{a_1 b_2 - b_1 a_2}{b_1 - a_1}$$

$$\downarrow$$

$$(b_2 - a_2)x - (b_1 - a_1)y = a_1 b_2 - b_1 a_2.$$

(or $v_2 x - v_1 y = a_1 v_2 - a_2 v_1$ if you're given the direction)

This is not the case for the 3D lines, because we have two many equations. Still, we can remove t as follows.

$$x = (b_1 - a_1)t + a_1 \qquad y = (b_2 - a_2)t + a_2 \qquad z = (b_3 - a_3)t + a_3$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$t = \frac{x - a_1}{b_1 - a_1} \qquad t = \frac{y - a_2}{b_2 - a_2} \qquad t = \frac{z - a_3}{b_3 - a_3}$$

so we get

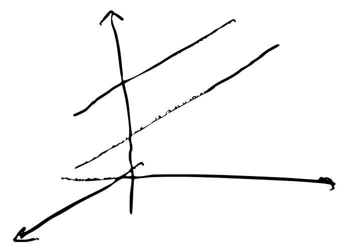
$$(t=)\ \boxed{\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}}$$

(OR $\boxed{\frac{x-a_1}{v_1} = \frac{y-a_2}{v_2} = \frac{z-a_3}{v_3}}$ if you're given the direction)

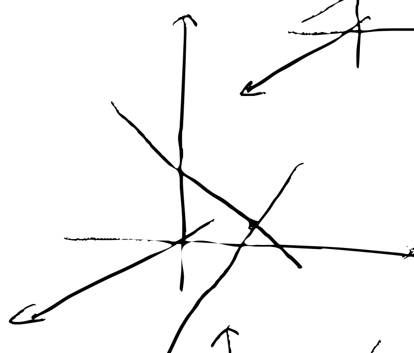
these are called the symmetric equations of a line.

Given two lines, three things can happen.

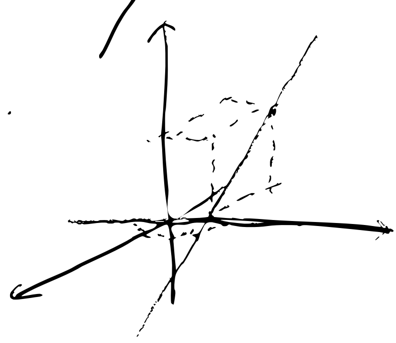
- They are parallel to each other.



- They intersect.



- Neither of the above happens.
  They are skew.



Given two lines $x = a_1 + v_1 t$ and $x = b_1 + w_1 t$

$$y = a_2 + v_2 t \qquad y = b_2 + w_2 t$$

$$z = a_3 + v_3 t \qquad z = b_3 + w_3 t,$$

- They are parallel if the direction vectors $\langle v_1, v_2, v_3 \rangle$ and $\langle w_1, v_2, w_3 \rangle$ are parallel.

- They intersect if there are values of s, t such that

$$a_1 + v_1 t = x = b_1 + w_1 s$$
$$a_2 + v_2 t = y = b_2 + w_2 s$$
$$a_3 + v_3 t = z = b_3 + w_3 s.$$

(Solving a system of linear equations)

Here we use two different variables. This is because the two parametrizations can be very different, so the timing of the particles may not match.

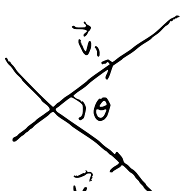
Example

$$x = t \qquad x = t+1$$
$$y = t \qquad y = t+1$$
$$z = t \qquad z = t+1$$

represent the same line, but the particles never coincide.

- They are skew if none of the above holds

More generally, we can refer to the angle between the two direction vectors as the acute angle between the two angles



$$\cos\theta = \left| \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|} \right| \quad \text{(dot product!)}$$

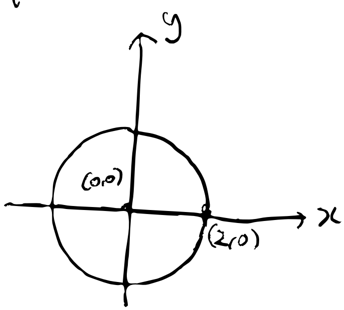
This is because θ never exceeds $\frac{\pi}{2}$ in this case.

Circles and ellipses

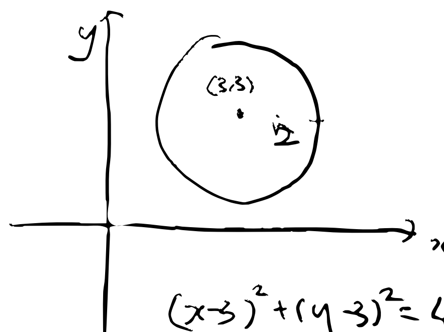
We first think about circles. The (implicit) equation for the circle centered at the origin $(0,0)$ with radius r is

$$x^2+y^2=r^2$$

If we want to move the center to (c,d), the equation becomes $(x-c)^2+(y-d)^2=r^2$.

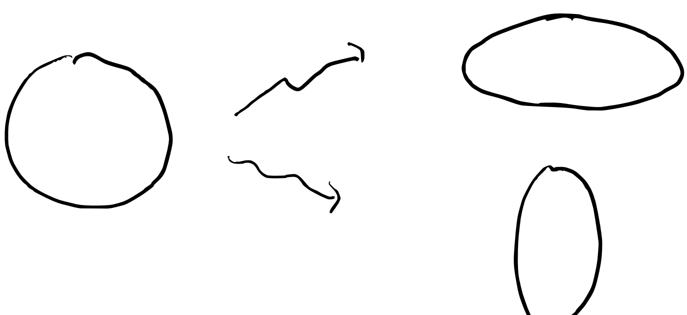


$$x^2+y^2=4$$



$$(x-3)^2+(y-3)^2=4.$$

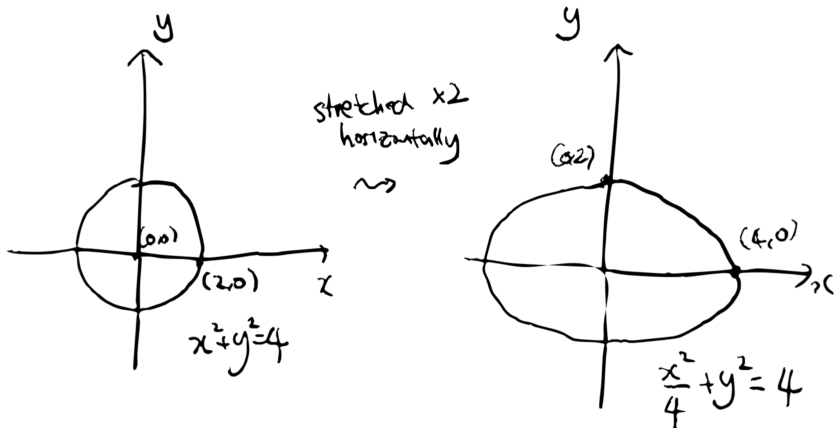
Ellipses are "squashed and stretched" circles.



A basic form of the equation for an ellipse is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2}$$

This is the equation for the ellipse that is stretched from the circle $x^2+y^2=r^2$ by $\times a$ in the x-direction and $\times b$ in the y-direction.



You can also move the center to (c,d), which will be expressed by

$$\boxed{\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = r^2.}$$

<u>Note</u> Circles are examples of ellipses. Indeed, $x^2+y^2=r^2$ is the same as $\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$, which means it's stretched out $\times r$ in both directions from the circle $x^2+y^2=1$.

We also now know that $x^2+y^2=r^2$ has a parametric equation

$$\boxed{\begin{array}{l} x = r\cos t \\ y = r\sin t. \end{array}}$$

So the center-shifted circle $(x-c)^2+(y-d)^2=r^2$ would have a parametric equation

$$\boxed{\begin{aligned} x &= r\cos t + c \\ y &= r\sin t + d \end{aligned}}$$

Also, the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2$ would have a parametric equation

$$\boxed{\begin{aligned} x &= ar\cos t \\ y &= br\sin t. \end{aligned}}$$

Finally, the center-shifted ellipse $\frac{(x-c)^2}{a^2}+\frac{(y-d)^2}{b^2}=r^2$ would have a parametric equation

$$\boxed{\begin{aligned} x &= ar\cos t + c \\ y &= br\sin t + d. \end{aligned}}$$

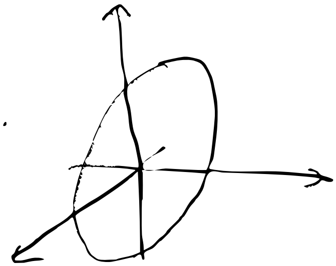
__Exercise__ Check the above parametric equations satisfy the implicit equations.

Any ellipse is a squashed version of the circle $x^2+y^2=1$, so you can imagine any ellipse is given by the distortion of x-axis and the y-axis on the circle. $\langle x,y\rangle = \vec{v}_1\cos t+\vec{v}_2\sin t$

So from the center (c,d), the position of a particle on an ellipse can be given by the vector function

$$\vec{v}_1 \cos t + \vec{v}_2 \sin t.$$

This also applies to the ellipses in 3D.



Example $\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = r^2$, or

$x = ar\cos t + c$, $y = br\sin t + d$ can be also thought as

$$\underbrace{\langle x-c,\, y-d\rangle}_{\text{position of the particle with respect to the center}} = \underbrace{\langle ar, 0\rangle}_{\vec{v}_1} \cos t + \underbrace{\langle 0, br\rangle}_{\vec{v}_2} \sin t.$$